Mapping from 1/ *r* Hubbard model to Gross - Neveu model and exclusion statistics

1997 J. Phys. A: Math. Gen. 30 1495

(http://iopscience.iop.org/0305-4470/30/5/017)

J. Phys. A: Math. Gen. **30** (1997) 1495–1502. Printed in the UK PII: S0305-4470(97)78591-3

Mapping from $1/r$ Hubbard model to Gross–Neveu model and exclusion statistics

Yue-lin Shen and Mo-lin Ge

Theoretical Physics Division, Nankai Institute of Mathematics, Nankai University, Tianjin 300071, People's Rebublic of China

Received 3 October 1996

Abstract. By the particle–hole transformation, the $1/r$ Hubbard model at the finite on-site coupling energy is mapped to the Gross–Neveu model in the continuous limit, while the spectrum is given by the Bethe ansatz equations. We demonstrate that the model is an ideal gas with exclusion statistics, and give the statistical interactions in the algorithm of Bernard and Wu.

1. Introduction

Since Yang introduced [1] the Bethe ansatz to study the one-dimensional many-body problem with δ-function interactions, the quantum integrable systems, including the Hubbard model [2], Thirring model [3], Gross–Neveu model [4] and Kondo problem [5], have been intensively studied. Recently, Haldane [6] and Shastry [7] introduced another integrable model, the $1/r^2$ Heisenberg chain, from which Haldane proposed the concept of exclusion statistics [8], a generalization of that Pauli principle. Bernard and Wu [9] pointed out that Bethe ansatz solvable models can be regarded as an ideal gas with exclusion statistics. A review article on this topic has appeared [10].

There have appeared many papers on the long-range models in the spirit of Haldane and Shastry. One of them is the $1/r$ Hubbard model suggested by Gebhard and Ruckenstein [11], who related the model to the Haldane–Shastry model in the large-U case and, through numerical diagonalization, found that the model exhibits a metal–insulator transition at $U = 2\pi t$, where U is the on-site energy and t is the hopping coefficient. However, that the result in large U is prolonged to $U \ll 2\pi t$ is not natural, although they claimed that the result can be compared with some limited cases. In [12], Wang and his co-workers found the Gutzwiller–Jastrow wavefunction for the model in the strong limit $U = \infty$. Obviously, to explore the model more clearly, we need exact results because of the invalidity of the conventional perturbative method. In [13], we found a lot of eigenstates via the η-pairing mechanism [14]. In this paper, we will develop the Bethe ansatz solution in the continuous limit by mapping the model to the well known Gross–Neveu model when the on-site coupling energy is finite, and point out that the model shares exclusion statistics.

The paper is organized as follows. In section 2, by mapping the $1/r$ Hubbard model to the Gross–Neveu model, we solve the model in terms of the Bethe ansatz; in section 3, we solve the energy spectrum in the case of given numbers of spin-up and spin-down particles in the thermodynamics limit; in section 4, we point out that the model can be described as an ideal gas with exclusion statistics in the algorithm of Bernard and Wu; and, finally, we give some discussions in section 5.

2. Bethe ansatz

Let us start with the Hamiltonian of the model [11]

$$H = \sum_{\sigma=\pm, i\neq j} t_{ij} c^\dagger_{i\sigma} c_{j\sigma} + U \sum_i c^\dagger_{i\uparrow} c_{i\uparrow} c^\dagger_{i\downarrow} c_{i\downarrow} \tag{1}$$

with the long-range hopping

$$\begin{aligned} t_{ij} &= it(-)^{i-j}[d(i-j)]^{-1} \\ d(i-j) &= \frac{L}{a\pi} \sin\frac{\pi(i-j)a}{L} \end{aligned} \tag{2}$$

where a is the lattice spacing and L the length of the lattice. When we rescale a as the unit, L becomes the number of the lattice. We keep a to go through the continuous limit. Further we assume t and U positive and L even without any speciality.

The Fourier transformation of the kinetic term in H leads to

$$H = -t \sum_{k,\sigma=\pm} k c^\dagger_{k\sigma} c_{k\sigma} + U \sum_i c^\dagger_{i\uparrow} c_{i\uparrow} c^\dagger_{i\downarrow} c_{i\downarrow} \tag{3}$$

where

$$-\frac{\pi(L-1)}{La} \leqslant k \leqslant \frac{\pi(L-1)}{La}.$$

If we make the particle–hole transformation

$$c_{k\uparrow} = a_k \qquad c_{k\downarrow} = b^\dagger_k \qquad c_{i\uparrow} = a_i \qquad c_{i\downarrow} = b^\dagger_i. \tag{4}$$

Here we must note that the Fourier transformations for a and b are different, i.e if we define $a_l = \sqrt{1/L}\sum_k \exp(\mathrm{i}kl)a_k$ correspondingly we have $b_l = \sqrt{1/L}\sum_k \exp(-\mathrm{i}kl)b_k$, which obeys the Fourier transformations of $c_\uparrow$ and $c^\dagger_\downarrow$, respectively.

Through this process the Hamiltonian is recast into

$$\begin{aligned} H &= -t\sum_k k(a^\dagger_k a_k + b_k b^\dagger_k) + U\sum_i a^\dagger_i a_i b_i b^\dagger_i \\ &= -t\sum_k k(a^\dagger_k a_k - b^\dagger_k b_k) - U\sum_i a^\dagger_i a_i b^\dagger_i b_i + U\sum_i a^\dagger_i a_i. \end{aligned} \tag{5}$$

From transformation (4) we have different definitions of the vacumn for the spin-up and spin-down excitations. The vacuum for the spin-up excitations is just the usual empty state, with $a^\dagger_k|0\rangle$ as the electron; while the vacuum for the spin-down excitations is the filled Fermi sea by the down electrons, thus $b^\dagger_k|0\rangle$ is the annihilation of the electron in the Fermi sea, or creation of a hole. However, we will not care about the difference between them in what follows since they are equal in algebra.

As $\sum_i a^\dagger_i a_i = N^+$ is the conserved number of the spin-up electrons, we first consider

$$H_1 = -t\sum_k (a^\dagger_k a_k - b^\dagger_k b_k) - U\sum_i a^\dagger_i b^\dagger_i b_i a_i. \tag{6}$$

We should notice that *this procedure is invalid for the case of* $U = \infty$, when the double occupancy is forbidden, so therefore there is neither the coupling term nor the term UN^+.

We treat H_1 in the continuous limit as that done in the Kondo problem [5]. In general, the continuous limit is not valid in any case because of the short-distance fluctuation, and the coupling constants need renormalization if the scaling law is preserved. However, renormalization in the strong correlated system is not easy to deal with, so therefore what we use is prolongation as is done in [11]. We assume that the result in the weak coupling region can be prolonged to the strong coupling region, which is based on two important facts: the first is the duality relation between U and $2\pi t$; the second is the fact that the Bethe ansatz only concentrates on the lowest states, when the momentum is always restricted in the first Brillouin zone. Since there are not enough exact conclusions in the literature, we believe our results can shed some light on the study of this model.

The meaning of the continuous limit is to take $a \to 0$ while preserving L, when the momentum distribution will be over the total real axis, and the summation in (6) will be replaced by integration. Hence the Hamiltonian H_1 reads

$$H_1 = -t \int \mathrm{d}k\, k(a_k^\dagger a_k - b_k^\dagger b_k) - U \int \mathrm{d}x\, a_x^\dagger b_x^\dagger b_x a_x. \tag{7}$$

We define the field operators $\psi_+(x)$, $\psi_-(x)$ with the Fourier transformation relation

$$\begin{aligned} \psi_+(x) &= a_x = \frac{1}{\sqrt{L}} \int \mathrm{d}k\, \mathrm{e}^{\mathrm{i}kx} a_k \\ \psi_-(x) &= b_x = \frac{1}{\sqrt{L}} \int \mathrm{d}k\, \mathrm{e}^{\mathrm{i}kx} b_k \end{aligned} \tag{8}$$

and the Hamiltonian H_1 can be rewritten in the second quantization form

$$H_1 = \mathrm{i}t \sum_{a=\pm} \int \alpha_a \psi_a^\dagger \partial_x \psi_a(x) - U \sum_{a<b} \int \mathrm{d}x\, \psi_a^\dagger(x)\psi_a(x)\psi_b^\dagger(x)\psi_b(x). \tag{9}$$

Here we have defined the chirality

$$\alpha_a = \begin{cases} +1 & a = + \\ -1 & a = -1. \end{cases}$$

Until now, we found that the lattice model is canonically mapped to the Gross–Neveu model in the continuous limit, and hence the most general solution for (9) is assumed [5] to be

$$|\Phi\rangle = \sum_{a_i} \int \prod_{i=1}^{N} \mathrm{d}x_i \phi(x_1 a_1, x_2 a_2, \ldots, x_N a_N) \prod_{i=1}^{N} \psi_{a_i}^\dagger(x_i)|0\rangle \tag{10}$$

where the physical vacuum $|0\rangle$ is defined by $\psi_a(x)|0\rangle = 0$, and a_i is the spin of the particle on site x_i. In order for $|\Phi\rangle$ to be the eigenstate of H_1, ϕ must be an eigenstate of the following N-particle Hamiltonian,

$$h_1 = \mathrm{i}t \sum_i \alpha_i \partial_i - \frac{U}{4} \sum_{i<j} \delta(x_i - x_j)(\alpha_i - \alpha_j)^2 P_{ij}^{\mathrm{s}} \tag{11}$$

where α_i is the chirality of the particle at x_i, and the spin exchange operator P_{ij}^{s} is defined by

$$P_{ij}^{\mathrm{s}} \phi(\ldots a_i, \ldots a_j \ldots) = \phi(\ldots a_j \ldots a_i).$$

The antisymmetry of the total wavefunction requires that

$$\phi(\ldots (xa)_i \ldots (xa)_j \ldots) = (-)\phi(\ldots (xa)_j \ldots (xa)_i).$$

Hence, we have

$$h_1 = \mathrm{i}t \sum_i \alpha_i \partial_i + \frac{U}{4} \sum_{i<j} \delta(x_i - x_j)(\alpha_i - \alpha_j)^2 P_{ij} \tag{12}$$

where P_{ij} only exchanges the positions of the particles. From (12), we can assume the following wavefunction with N particles labelled by momentums $k_1 \ldots k_N$ and spins $a_1 \ldots a_N$.

$$\phi(x,a) = \sum_{Q,P\in S_N} A_{QP}\theta(x_Q)\exp\left[\mathrm{i}\sum_j k_{Pj}x_{Qj}\right]\prod_l \delta^{a_{Pl}}_{a_{Ql}} \tag{13}$$

where Q, P label the different configurational regions with $\theta(x_Q)$ referring to $0 \leqslant x_{Q1} \leqslant \cdots \leqslant x_{QN} \leqslant L$. The corresponding energy and momentum for the state of (13) are

$$E_1 = -t\sum_{i=1}^{N}\alpha_i k_i$$
$$P = \sum_{i=1}^{N} k_i. \tag{14}$$

The periodic boundary condition requires certain eigenequations for A_{QP}, which can be solved by the generalized Bethe ansatz [1–5]. The final result is the auxiliary equations for k,

$$\mathrm{e}^{\mathrm{i}k_a L} = \prod_{\alpha_b\neq\alpha_a, b=1}^{N}\frac{r^2-1+\mathrm{i}(\alpha_b-\alpha_a)r}{r^2+1}\prod_{j=1}^{N^-}\frac{\mathrm{i}(\alpha_a-\lambda_j)+c/2}{\mathrm{i}(\alpha_a-\lambda_j)-c/2}$$
$$\prod_{b=1}^{N}\frac{\mathrm{i}(\alpha_b-\lambda_j)+c/2}{\mathrm{i}(\alpha_b-\lambda_j)-c/2} = -\prod_{k=1}^{N^-}\frac{\mathrm{i}(\lambda_j-\lambda_k)-c}{\mathrm{i}(\lambda_j-\lambda_k)+c} \tag{15}$$

where N^- is the number of the spin-down electrons, N is the total number of electrons, λ_j is named rapidity or the momentum of the hole and $c = (r^2-1)/r$ with $r = 4t/U$.

To get an insight into (15), we take the logarithm of (15) which yields

$$N^+\theta(2\lambda_j - 2) + N^-\theta(2\lambda_j + 2) = \sum_{k=1}^{N^-}\theta(\lambda_j - \lambda_k) + 2\pi J_j$$
$$k_a = \frac{2\pi}{L}n_a + \frac{1}{L}\sum_{j=1}^{N^-}\theta(2\lambda_j - 2\alpha_a) - \frac{1}{4L}[(1-\alpha_a)N^+ - (1+\alpha_a)N^-]\theta_0. \tag{16}$$

Here we have used the definition $\theta(x) = -2\,\mathrm{arctg}(x/c), -\pi \leqslant \theta \leqslant \pi$, whereas $\theta_0 = 2\arccos(r^2-1)/(r^2+1), 0 \leqslant \theta_0 \leqslant 2\pi$. We have defined θ_0 as an arccosine function instead of arctangenta function since the phase factor of $[r^2-1+\mathrm{i}(\alpha_b-\alpha_a)r]/(r^2+1)$ does not change when r turns from 1^- to 1^+. If we define it in the arctangenta function, it will cause a π ambiguity when we take the logarithm of this factor. The problem also comes into the other factors in (15), but when we take the logarithm of them, they only cause a 2π ambiguity which has no effect, because we have to add these terms to two quantum numbers n_a and J_k. In (16) the two quantum numbers n_a, J_k are given according to N and N^-:

(1) N even, N^- odd

n_a takes ascending half integers, $-(N-1)/2$ to $(N-1)/2$

J_k takes ascending integers, $-(N^- - 1)/2$ to $(N^- - 1)/2$

(2) N even, N^- even

n_a takes ascending integers, $-N/2$ to $N/2$

J_k takes ascending half integers, $-(N^- - 1)/2$ to $(N^- - 1)/2$

(3) N odd, N^- even
n_a takes ascending half integers, $-(N-1)/2$ to $(N-1)/2$
J_k takes ascending integers, $-N^-/2$ to $N^-/2$
(4) N odd, N^- odd
n_a takes ascending half integers, $-N/2$ to $N/2$
J_k takes half integers, $-N^-/2$ to $N^-/2$.

So far we have produced equations for the momentum distributions; however, it is never easy to solve (16). In the next section, we solve it in the thermodynamic limit.

3. Spectrum in the thermodynamic limit

In this section, we shall derive the spectrum in the thermodynamic limit. It is usually accepted that the Bethe ansatz solution gives the ground state for given numbers of the spin-up and spin-down electrons.

We note that the energy for (1) should be obtained from $E = E_1 + UN^+$, hence

$$E = -t\sum_{i=1}^{N}\alpha_i k_i + UN^+.$$

However, since N^+ is a constant and does not affect the spectrum structure, we neglect it and first discuss E_1, which reads from (16)

$$\begin{aligned}E_1 = -t\sum_{i=1}^{N}\alpha_i k_i &= -t\sum_{i}^{N}\alpha_i\left[\frac{2\pi}{L}n_i + \frac{1}{L}\sum_{j=1}^{N^-}\theta(2\lambda_j - 2\alpha_i)\right.\\ &\qquad\left. -\frac{1}{4L}((1-\alpha_i)N^+ - (1+\alpha_i)N^-)\theta_0\right]\\ &= -\frac{2\pi t}{L}\sum_{i}^{N}\alpha_i n_i - \frac{tN^+}{L}\sum_{j}^{N^-}\theta(2\lambda_j - 2)\\ &\qquad + \frac{tN^-}{L}\sum_{j=1}^{N^-}\theta(2\lambda_j + 2) - \frac{tN^+N^-}{L}\theta_0.\end{aligned} \tag{17}$$

The meaning of thermodynamic limit is $N^+ \to \infty$, $N^- \to \infty$, $L \to \infty$ with N^+/L and N^-/L kept constant. If we define $\sigma(\lambda_\gamma) = 1/(\lambda_{\gamma+1} - \lambda_\gamma)$ when $J_{\gamma+1} = J_\gamma + 1$, then from (16) and (17) we obtain

$$\begin{aligned}E_1 = -\frac{2\pi t}{L}\sum_{i}^{N}\alpha_i n_i - \frac{tN^+}{L}\int \mathrm{d}\lambda\,\sigma(\lambda)\theta(2\lambda - 2)\\ + \frac{tN^-}{L}\int \mathrm{d}\lambda\,\sigma(\lambda)\theta(2\lambda + 2) - \frac{tN^+N^-}{L}\theta_0\end{aligned} \tag{18}$$

$$N^+\theta(2\lambda - 2) + N^-\theta(2\lambda + 2) = \int \mathrm{d}\lambda'\,\sigma(\lambda')\theta(\lambda - \lambda') + 2\pi J_\lambda. \tag{19}$$

Differentiating (19) with respect to λ, we obtain

$$\sigma(\lambda) = f(\lambda) - \int k(\lambda - \lambda')\sigma(\lambda')\,\mathrm{d}\lambda' \tag{20}$$

where

$$f(\lambda) = \frac{2c}{\pi}\left[\frac{N^+}{c^2 + 4(\lambda - 1)^2} + \frac{N^-}{c^2 + 4(\lambda + 1)^2}\right]$$

$$k(\lambda-\lambda')=\frac{c}{\pi}\frac{1}{c^2+(\lambda-\lambda')^2}. \tag{21}$$

The solution of (20) can be obtained by the Winer–Hopf technique developed by Yang and Yang [15]; it reads

$$\sigma(\lambda)=\frac{1}{2c}\left[\frac{N^+}{ch(\pi/c)(\lambda-1)}+\frac{N^-}{ch(\pi c)(\lambda+1)}\right]. \tag{22}$$

Hence, we have

$$E_1=-\frac{2\pi t}{L}\sum_i^N\alpha_i n_i+\frac{t}{2cL}\int \mathrm{d}\lambda\left[\frac{N^+}{ch(\pi/c)(\lambda-1)}+\frac{N^-}{ch(\pi/c)(\lambda+1)}\right]$$
$$\times[N^-\theta(2\lambda+2)-N^+\theta(2\lambda-2)]-\frac{tN^+N^-}{L}\theta_0. \tag{23}$$

To obtain the ground state, we must minimize the term $\sum_{i=1}^N\alpha_i n_i$ in (23). Using the permitted value of n_i, for example, N, N^- even, we should choose

$$n_i=\frac{N}{2},\frac{N}{2}-1,\ldots,\frac{N}{2}-(N^+-1)\qquad \text{for } \alpha_i=1$$

and

$$n_i=-\frac{N}{2},-\frac{N}{2}+1,\ldots,-\frac{N}{2}+(N^--1)\qquad \text{for } \alpha_i=-1$$

then we have

$$\sum_{i=1}^N\alpha_i N_i=N^+N^-+\frac{1}{2}(N^++N^-).$$

Hence the energy of the ground state for given N^+ and N^- is

$$E_g=E_{1g}+UN^+=-\frac{2\pi t}{L}\left(N^+N^-+\frac{N^+}{2}+\frac{N^-}{2}\right)$$
$$+\frac{t}{2cL}\int \mathrm{d}\lambda\left[\frac{N^+}{ch(\pi/c)(\lambda-1)}+\frac{N^-}{ch(\pi/c)(\lambda+1)}\right]$$
$$\times[N^-\theta(2\lambda+2)-N^+\theta(2\lambda-2)]-\frac{tN^+N^-}{L}\theta_0+UN^+. \tag{24}$$

It is interesting to ask how to compare the states from Bethe ansatz with those in [11, 12]. We claim that neither their results in large U limit nor our Bethe ansatz result has revealed all the properties in the model. Our discussion is valid when U is finite, whereas their results are valid when U is large.

4. Exclusion statistics

Exclusion statistics is a generalization of the Pauli principle proposed by Haldane [8], and its basic idea is state counting. Consider the Hilbert space H_α of a single particle of specie α, confined to a region of matter. In general, the dimension d_α will change as particles are added, so Haldane defined the statistical interaction $\alpha_{\alpha\beta}$ through the difference relation

$$\Delta d_\alpha=-\sum_\beta\alpha_{\alpha\beta}\Delta N_\beta \tag{25}$$

where $\{\Delta N_\beta\}$ is a set of allowed changes of the particle numbers at fixed size and boundary condition. Until now, many physical systems can be viewed as an ideal gas with exclusion statistics, see the review article [10]. Bernard and Wu described the Bethe ansatz solvable model as belonging to this category [9], and Wu [10] generalized this concept to the models with internal freedoms. It is well known that there only remains some auxiliary equations to solve in the Bethe ansatz

$$LP_\mu(\lambda_i^\mu) = 2\pi I_i^\mu + \sum_{j\nu} \theta_{\mu\nu}(\lambda_i^\mu, \lambda_j^\nu). \tag{26}$$

Here μ and ν label different kinds of quasiparticles or excitations, i and j different roots of the Bethe ansatz equations. The pseudomomentum $p_\mu(\lambda_i^\mu)$ is a certain given function of the rapidity λ_i^μ, and $\theta_{\mu\nu}(\lambda_i^\mu, \lambda_j^\nu)$ is the two-body scattering phase shift between two quasiparticles with rapidity λ_i^μ and λ_j^ν. Again, L is the size of the system, and $\{I_i^\mu\}$ is a set of integers or half-integers satisfying $I_{i+1}^\mu > I_i^\mu$. In the thermodynamic limit, the above equation becomes

$$\rho_\mu^0 = \frac{1}{2\pi} P'_\mu + \sum_\nu \int \alpha_{\mu\nu}(\lambda, \lambda')\rho_\nu(\lambda')\,\mathrm{d}\lambda'. \tag{27}$$

The statistical interaction may be read from (26), but can be specified in a given model: Wu [10] defined

$$\alpha_{\mu\nu}(\lambda, \lambda') = \delta_{\mu\nu}\delta(\lambda - \lambda') + \frac{1}{2\pi}\frac{\partial}{\partial\lambda}\theta_{\mu\nu}(\lambda, \lambda'). \tag{28}$$

In our case, we need a little modification. First, let us rewrite (16) as

$$\begin{aligned}
&\sum_{a=1}^{N} \theta(2\lambda_j - 2\alpha_a) = \sum_{k=1}^{N^-} \theta(\lambda_j - \lambda_k) + 2\pi J_j \\
&Lk_a = 2\pi n_a + \sum_{j=1}^{N^-} \theta(2\lambda_j - 2\alpha_a) + \sum_{b\neq a} \theta_0(\alpha_b - \alpha_a)
\end{aligned} \tag{29}$$

where $\theta_0(\alpha_a - \alpha_b) = \frac{1}{2}(\alpha_a - \alpha_b)\arccos(r^2 - 1)/(r^2 + 1)$.

Considering the summation over the discrete variables and the integration over the continuous variables, we write down

$$\begin{aligned}
&\theta_{\mathrm{cc}}(k_a\alpha_a, k_b\alpha_b) = -\theta_0(\alpha_a - \alpha_b) \\
&\theta_{\mathrm{cs}}(k_a\alpha_a, \lambda_j) = \theta(2\lambda_j - 2\alpha_a) \\
&\theta_{\mathrm{ss}}(\lambda_j, \lambda_k) = -\theta(\lambda_j - \lambda_k) \\
&\theta_{\mathrm{sc}}(\lambda_j, k_a\alpha_a) = -\theta(2\lambda_j - 2\alpha_a)
\end{aligned} \tag{30}$$

where c refers to the electron or charge excitation, and s refers to the hole or spinon. From Wu's algorithm, we have the statistical interactions

$$\begin{aligned}
&\alpha_{\mathrm{cc}}(k\alpha_k, k'\alpha_{k'}) = \delta_{\alpha_k\alpha_{k'}}\delta(k - k') \\
&\alpha_{\mathrm{sc}}(\lambda, k\alpha_k) = \frac{2c}{\pi}\frac{1}{c^2 + 4(\lambda - \alpha_k)^2} \\
&\alpha_{\mathrm{ss}}(\lambda, \lambda') = \delta(\lambda - \lambda') + \frac{1}{\pi}\frac{c}{(\lambda - \lambda')^2 + c^2} \\
&\alpha_{\mathrm{cs}}(k\alpha_k, \lambda) = 0.
\end{aligned} \tag{31}$$

The result of the above equation is derived from the continous form of the Bethe equations. We observe that the $1/r$ Hubbard model can be viewed as a generalized ideal gas with the statistical interaction among the spin-up, spin-down electrons and spinons.

5. Discussion

Based on the Bethe ansatz solution, we have produced the ground state and the ideal gas description of the $1/r$ Hubbard model. In [11], it was pointed out that this model is related to the $1/r^2$ Heisenberg chain. Now we see that the two models share some common characters, such as exact solvability and the ideal gas description.

An open question is whether the model belongs to the Yang–Baxter system. Recently the Yangian symmetry has been set up in this model [20]; however, this does not mean integrability, so a thoughtful discussion on this problem is desirable.

Acknowledgment

The authors would like to thank the beneficial discussions with Prof Y S Wu.

References

[1] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
[2] Lieb E H and Wu F Y 1968 *Phys. Rev. Lett.* **20** 1445
[3] Bergknoff H and Thacker H B 1979 *Phys. Rev. Lett.* **42** 125
Bergknoff H and Thacker H B 1980 *Phys. Rev.* D **19** 3666
[4] Andrei N and Lowenstein J H 1979 *Phys. Rev. Lett.* **43** 1698
[5] Andrei N 1980 *Phys. Rev. Lett.* **45** 379
[6] Haldane F 1988 *Phys. Rev. Lett.* **60** 635
[7] Shastry B 1988 *Phys. Rev. Lett.* **60** 639
[8] Haldane F 1991 *Phys. Rev. Lett.* **67** 937
[9] Bernard D and Wu Y S 1995 *Yangian and Recent Development on the Quantum Integrable Models* ed M L Ge and Y S Wu (Singapore: World Scientifics)
[10] Wu Y S 1995 Fractional statistics and generalized ideal gas *Utah Preprint*
[11] Gebhard F and Ruckenstein A E 1992 *Phys. Rev. Lett.* **68** 244
Gebhard F and Ruckenstein A E 1994 *Phys. Rev.* B **49** 10 926
[12] Wang D F, Zhong Q F and Coleman P 1993 *Phys. Rev.* B **46** 9395
[13] Shen Y L and Ge M L 1995 *Phys. Rev.* B **52** 11 667
[14] Yang C N 1989 *Phys. Rev. Lett.* **63** 2144
Yang C N and Zhang S C 1990 *Mod. Phys. Lett* **B4** 759
[15] Yang C N and Yang C P 1962 *Phys. Rev.* **125** 164
[16] Lieb E H and Mattis D C 1962 *Phys. Rev.* **125** 164
[17] Gebhard F and Girndt A 1994 *Z. Phys* B **93** 455
[18] Dzierzawa M, Baeriswyl D and Stasio M 1995 *Phys. Rev.* **51** 1993
[19] Haldane F, Ha Z, Talstra J, Bernard D and Pasquier V 1992 *Phys. Rev. Lett.* **69** 2021
[20] Gohmann F and Inozemtser V 1996 *Phys. Lett.* **214A** 161